Rank reversal phenomenon in cross-efficiency evaluation of data envelopment analysis

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Keywords
Multiple Criteria Decision Making (MCDM), Rank reversal, and Least Common Multiple (LCM)

Abstract
This paper presents that the rank reversal occurs in other popular MCDM approach as well, cross-efficiency evaluation of data envelopment analysis (DEA), which has been alternative method for ranking decision making units (DMU) in the data envelopment analysis (DEA). This paper also attempts to illustrate that the proposed least common multiple (LCM) approach successfully addresses these rank reversal problems in decision support systems area.

1. Introduction

Analytic hierarchy process (AHP) (Saaty, 2009) has become a very popular multiple criteria decision making (MCDM) technique. AHP has been applied to diverse fields of study such as software evaluation, manufacturing systems, organizational performance evaluation, customer requirement rating, and financial industries. However, for nearly the same duration, AHP has also been criticized for rank reversals when a decision alternative is added or dropped, first noted by Belton & Gear (1983).

In order to avoid rank reversal phenomenon in the AHP where such reversals should not take place, many other different mathematical approaches (Dyer, 1985; Schoner & Wedley & Choo, 1993; Barzilai & Golany, 1994; Lootsma, 1999) have been proposed. It is noticeable that none of these methods have resolved this problematic phenomenon and there are still on-going debates on how to avoid rank reversals.

Rank reversal is also found in other popular MCDM approach as well, such as the cross-efficiency evaluation of data envelopment analysis (DEA). This paper illustrates that the rank reversals occur in other MCDM method and presents that the proposed method successfully addresses these rank reversal problems in decision support systems area.

2. Rank reversal in the cross-efficiency evaluation of data envelopment analysis (DEA)

There are a wide range of MCDM problem solution techniques, varying in complexity and possible solutions. Each method has its own strength, weaknesses and possibilities to be applied. For example, Borda-Kendall (BK) method is the most widely used tool in determining a consensus ranking because of its computational simplicity. It uses a weighted ordinal ranking model in which each of a set of *n* alternatives was given an ordinal rank on a set of criteria. Simple Additive Weighting (SAW) is simple and the most frequently used multiple attribute decision making (MADM) tool. However, it is obvious that these popular MCDM approaches also suffer from rank reversal. In this section we illustrate that the rank reversals occur in cross-efficiency evaluation of data envelopment analysis (DEA) MCDM approach.

Cross-efficiency evaluation (Doyle & Green, 1994; Sexton and Silkman, 1986) has been alternative method for ranking decision making units (DMU) in the data envelopment analysis (DEA) (Chrnes & Cooper, 1978). It uses self or peer evaluations for performance assessment of DMUs. Consider n DMUs that are to be evaluated in terms of m inputs and s outputs. Let x_{ij} ($i = 1,\ldots, m$) and y_{ij} ($r = 1,\ldots, s$) be the input and output values of DMU_j ($j = 1,\ldots, n$). Then, the efficiencies of the n DMUs can be written as $\theta_j = \sum_{r=1}^{s} u_r y_{rj} \;/ \sum_{i=1}^{m} v_i x_{ij}$, $j = 1, n$. where v_i ($i =$

1,…, m) and u_r (*r = 1,…, s*) are input and output weights. For a specific DMU_k, $k \in \{1, \ldots, n\}$, its efficiency reltive to the other DMUs can be measured by the following CCR model [3]:

Maximize $\theta_{kk} = \sum_{r=1}^{s} u_{rk} y_{rk}$ (4)

Subject to $\sum_{i=1}^{m} v_{ik} x_{ik} = 1$, and

$\sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0$,

where $j = 1, \ldots, n$, $u_{rk} \geq 0$, $r = 1, \ldots, s$, and $v_{ik} \geq 0$, $i = 1, \ldots, m$.

Let u'_{rk} and v'_{ik} be an optimal solution to the above model (4). When θ'_{kk}=1 (where $\theta'_{kk} = \sum_{r=1}^{s} u'_{rk} y_{rk}$), DMU_k is referred to as the CCR-efficiency or simple efficiency of DMU_k, which is the best relative efficiency. if $\theta'_{kk} \neq 1$, it is referred to as non-DEA efficient. And $\theta_{jk} = \sum_{r=1}^{s} u'_{rk} y_{rj} \,/\, \sum_{i=1}^{m} v'_{ik} x_{ik}$ is referred to as cross-efficiency of DMU_j, which reflects the peer-evaluation of DMU_k to DMU_j ($j = 1, \ldots, n$; $j \neq k$).All DEA efficient units determine an efficient frontier.

CCR model [6] is computed for each DMU, individually. As a result, there are *n* sets of input and output weights for the *n* DMUs. Each DMU has (*n-1*) cross-efficiencies plus one CCR-efficiency (Table 1). Since this model may have multiple optimal solutions, this non-uniqueness could potentially hamper the use of cross-efficiency. To resolve this problem, Sexton & Silkman (1986) suggested the aggressive formulation for cross-efficiency evaluation, which minimizes the cross-efficiencies of the other DMUs to avoid the arbitrariness of cross-efficiency.

Minimize $\sum_{r=1}^{s} u_{rk} \left(\sum_{j=1, j\neq k}^{n} y_{rj}\right)$ (5)

Subject to $\sum_{i=1}^{m} v_{ik} \left(\sum_{j=1, j\neq k}^{n} x_{ij}\right) = 1$, and

$\sum_{r=1}^{s} u_{rk} y_{rk} - \theta'_{kk} \sum_{i=1}^{m} v_i x_{ij} = 0$,

$\sum_{r=1}^{s} u_{rk} y_{rj} - \sum_{i=1}^{m} v_{ik} x_{ij} \leq 0$,

where $j = 1, \ldots, n$; $j \neq k$, $u_{rk} \geq 0$, $r=1, \ldots, s$, and $v_{ik} \geq 0$, $i = 1, \ldots, m$, θ'_{kk} is the CCR-efficiency of DMU_k.

DMU	Target DMU				Average
	1	2	…	n	Cross-efficiency
1	θ_{11}	θ_{12}	…	θ_{1n}	$\frac{1}{n}\sum_{k=1}^{n}\theta_{1k}$
2	θ_{21}	θ_{22}	…	θ_{2n}	$\frac{1}{n}\sum_{k=1}^{n}\theta_{2k}$
:	:	:		:	:
n	θ_{n1}	θ_{n2}	…	θ_{nn}	$\frac{1}{n}\sum_{k=1}^{n}\theta_{nk}$

Table 1: Cross-efficiency matrix of n DMUs

Department (DMU)	Outputs			Inputs			CCR-efficiency
	y1	*y2*	*y3*	*x1*	*x2*	*x3*	
1	60	35	17	12	400	20	1
2	139	41	40	19	750	70	1
3	225	68	75	42	1500	70	1
4	90	12	17	15	600	100	0.8197
5	253	145	130	45	2000	250	1
6	132	45	45	19	730	50	1
7	305	159	97	41	2350	600	1

Table 2: Data matrix for seven departments in a university

However, it is found that the cross-efficiency evaluation also suffers from the rank reversal phenomenon when a non-DEA efficient unit is added or removed. Consider the example investigated by Wong & Beasley (2012). There are seven departments (DMUs) in a university to be evaluated in terms of three inputs and three outputs (Table 2), which are defined as follows:

x_1 : Number of academic staff
x_2 : Academic staff salaries in thousands of pounds
x_3 : Support staff salaries in thousands of pounds
y_1 : Number of undergraduate students
y_2 : Number of postgraduate students
y_3 : Number of research papers.

It is seen from the CCR-efficiencies in Table 2 that DMU_4 is the only department that is rated as non-DEA efficient and all the other six departments determine an efficient frontier. Table 3 shows the aggressive cross-efficiencies of the seven departments which are obtained by solving model (5), which aims to minimize the cross-efficiencies of the other DMUs. It is seen that DMU_4 is evaluated as the least efficient department and DMU_6 is the most efficient department.

It is reasonably expected that removal of DMU_4 has no impact on the efficiencies of the other six departments because this DMU_4 is a non-DEA efficient department and not on the efficient frontier. Now DMU_4 is removed from the set of DMUs. However, this removal is found to have a significant impact on the cross-efficiencies of the other six departments. Table 4 shows the aggressive cross-efficiencies of the other six departments after the removal of DMU_4. The ranking between DMU_1 and DMU_6 is reversed with DMU_1 becoming the best department after DMU_4 is removed from the set of DMUs. However, it also can be observed that the proposed LCM method preserves the original rankings with DMU_6 as the most efficient department.

Department (DMU)	Target DMU							Average cross-efficiency	Rank	LCM	Rank
	1	2	3	4	5	6	7				
1	1.000	0.845	0.933	0.687	0.645	0.793	0.752	0.808	2	0.189	3
2	0.335	1	0.618	1.000	0.824	0.701	0.556	0.719	4	0.191	2
3	0.555	0.848	1.000	0.735	0.813	1.000	0.418	0.767	3	0.166	5
4	0.069	0.755	0.280	0.820	0.367	0.236	0.206	0.390	7	0.121	7
5	0.331	0.662	0.315	0.765	1.000	0.699	0.831	0.658	5	0.168	4
6	0.514	1.000	0.821	0.951	1.000	1.000	0.611	0.842	1	0.202	1
7	0.151	0.604	0.158	1.000	0.525	0.246	1.000	0.526	6	0.137	6

Table 3: Aggressive cross-efficiencies of the seven departments

Perez & Jineno (2006) pointed out that the rank reversal in the AHP could also be caused by the addition or deletion of indifferent criteria. This is also true to the cross-efficiency evaluation. For convenience, we consider an input or output as unimportant if it makes no contribution to CCR-efficiency. When an unimportant input or output is added or removed, the cross-efficiency evaluation may also suffer from the rank reversal phenomenon.

Department (DMU)	Target DMU						Average cross-efficiency	Rank	LCM	Rank
	1	2	3	5	6	7				
1	1.000	0.845	0.933	0.645	0.933	0.752	0.851	1	0.170	3
2	0.335	1	0.618	0.824	0.843	0.556	0.696	4	0.172	2
3	0.555	0.848	1.000	0.813	1.000	0.418	0.772	3	0.150	5
5	0.331	0.662	0.315	1.000	0.478	0.831	0.603	5	0.151	4
6	0.514	1.000	0.821	1.000	1.000	0.611	0.824	2	0.183	1
7	0.151	0.604	0.158	0.525	0.278	1.000	0.453	6	0.123	6

Table 4: Aggressive cross-efficiencies of the six departments without DMU_4

As is known for most multiple comparison decision making problems, in order to get rid of the dimensions of different decision attributes, normalization is necessary. Examples of the rank reversal seem to depict that the rank reversal is presumably caused by procedural flaws of the normalization method. The alternative approach to yield the most reliable initial ranking and to preserve the ranking is proposed in next section.

3. A proposed Least Common Multiple (LCM) approach

Shin et al. [13] propose an alternative approach that converts all measurement values of alternatives to the commensurate values by multiplying a least common multiple (LCM) of all column sums of criteria in the decision matrix. Before the composite weights of all alternatives are computed, a matrix, A_{ij}' is multiplied by L, a least common multiple of all column sums of criteria, where

$$L = \sum_{j=1} \sum_{i=1} a_{ij} \tag{1}$$

Now the weight vector of criteria (C_j) is given by $C_j = [\ c_1 \quad c_2 \quad c_3 \ \text{------------} \ c_j\ ]^T$. Then, multiplying the criteria weight vector C_j by the revised value matrix A_{ij}'' yields the following data matrix, X_i.

$$X_i = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \cdots \\ x_i \end{bmatrix} = \begin{bmatrix} \frac{a_{11}\ c_1 L}{\sum_{i=1} a_{i1}} + \frac{a_{12}\ c_2 L}{\sum_{i=1} a_{i2}} + \frac{a_{13}\ c_3 L}{\sum_{i=1} a_{i3}} + \dots\dots\dots + \frac{a_{1j}\ c_j L}{\sum_{i=1} a_{ij}} \\ \frac{a_{21}\ c_1 L}{\sum_{i=1} a_{i1}} + \frac{a_{22}\ c_2 L}{\sum_{i=1} a_{i2}} + \frac{a_{23}\ c_3 L}{\sum_{i=1} a_{i3}} + \dots\dots\dots + \frac{a_{2j}\ c_j L}{\sum_{i=1} a_{ij}} \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \frac{a_{i1}\ c_1 L}{\sum_{i=1} a_{i1}} + \frac{a_{i2}\ c_2 L}{\sum_{i=1} a_{i2}} + \frac{a_{i3}\ c_3 L}{\sum_{i=1} a_{i3}} + \dots\dots\dots + \frac{a_{ij}\ c_j L}{\sum_{i=1} a_{ij}} \end{bmatrix} \tag{2}$$

Finally, the normalized composite weights of alternatives are obtained from the following equation,

$$X_i' = \left[\frac{x_1}{\sum_{i=1} x_i} \quad \frac{x_2}{\sum_{i=1} x_i} \quad \frac{x_3}{\sum_{i=1} x_i} \cdots\cdots \frac{x_i}{\sum_{i=1} x_i} \right]^T \tag{3}$$

Because of the converted matrix of the unified commensurate unit, rank reversal problems in the AHP can be prevented without adjusting the weights of criteria or wondering about structural or functional dependency and independency. To verify the validity of our proposed approach, the next paragraphs present the results of LCM mode by re-examining the decision matrices used in the cross-efficiency evaluation examples in the previous section.

Consider the same numerical example in Table 2. It is seen that output 3 has no contribution to the CCR-efficiencies of the seven departments. For an unimportant input or output, it can be removed from the set of input or output indices without any impact on the CCR-efficiencies. However, Table 5 shows that the ranking between DMU_1 and DMU_6 is reversed after output3 is removed from the set of outputs.

These rank reversal phenomena give rise to a question. That is whether an unimportant input or output should be involved in the cross-efficiency evaluation. However, the proposed LCM method provides a consistent ranking of an original set of alternatives and preserves the original rankings where either the least efficient DMU or an unimportant input or output is dropped out from the decision making data matrix. Additionally, regarding the ranking of the original set of DMUs, the proposed method yields DMU_2 as the second best efficient department, which is different from that computed by the Cross-efficiency method in Table 3.

Department (DMU)	Target DMU							Average cross-efficiency	Rank	LCM	Rank
	1	2	3	4	5	6	7				
1	1.000	0.845	0.933	0.688	1.000	0.933	0.752	0.879	1	0.610	3
2	0.335	1	0.618	1.000	0.702	0.843	0.556	0.722	4	0.619	2
3	0.555	0.848	1.000	0.735	0.555	1.000	0.418	0.730	3	0.502	6
4	0.069	0.755	0.280	0.820	0.242	0.441	0.206	0.402	7	0.439	7
5	0.331	0.662	0.315	0.765	1.000	0.478	0.831	0.626	5	0.515	4
6	0.514	1.000	0.821	0.951	0.792	1.000	0.611	0.813	2	0.625	1
7	0.151	0.604	0.158	0.999	0.985	0.278	1.000	0.597	6	0.513	5

Table 5: Aggressive cross-efficiencies of the seven departments without output3

4. Conclusion

It is assumed that the AHP is a powerful multi-criteria decision making method and will continue to be useful for many future cases as it has been in the past. Despite this widespread usage, the AHP still suffers from some theoretical disputes. Rank reversals are also found in many other well-known MCDM methods. Many studies argue that the rank reversal phenomenon is unpreventable when any MCDM method is applied.

As seen in the rankings in Table 3, another primary criticism of MCDM methods is that due to the differences among different techniques, inconsistent results are obtained when applied to the same decision problem. It is important that a good MCDM method must not yield the ranking reversals when an alternative is added or removed. Even though the proposed method does not suffer from those problems, it is more important that additional research in decision analysis is necessary to produce the reliable rankings one may trust.

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