Linear programming model of production process optimization: a case study

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Abstract
This paper focuses on a current topic of production management and operations research which serves as a tool for small and medium enterprises to cope with pressure put on the by continuously changing market conditions and global economy itself. Firstly the paper presents a linear programming model developed in order to achieve a complex optimization of production process. Created model is general in its nature and therefore it can be applied on any kind of production process with just a few modifications. Therefore it is suitable for production enterprise with any type of process orientation. Model stresses the importance of cost minimization, which is also its main objective. Resource procurement costs, fixed and variable production costs, inventory costs and transportation costs are included. Proposed model also take into account market conditions. In the second part of the paper this general model is applied on a real-life production process in practice of selected production enterprise.

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Introduction
Nowadays production enterprises face a severe competition which puts that much pressure not only on their quality requirements, but also on effectiveness of their production processes. It is the goal of enterprise's operations management to ensure the best possible outcome and gain the competitive advantage which enables any enterprise to establish a desirable market position. However it is not a single set of managerial decisions which make it possible. A strive for excellence is a continuous process which does not only involve establishing a good market position, but it also focuses on implementing measures necessary to maintain it. Cost minimization is one of the original goals of all enterprises, which is nowadays viewed more as an essential part of enterprises' financial management. One of the newer ways enterprises can achieve excellence is through implementing specific measures in order to achieve flexibility of their processes. One of the basic tools is linear programming.

The main objective of this paper is to present the linear programming model of production process optimization whose main goal is to achieve cost minimization throughout the whole process. The body of paper is divided into three main parts. In the beginning we provide the motivation for this study which also includes a brief literature review of researched topics. We focus on process optimization and linear programming as an effective tool used in modelling. The next part of the paper describes the proposed model in detail. Firstly the general model is described and then this model is applied on a real-life production process. We use the example of this production enterprise as a case study of how the proposed general model can be modified specifically for the needs of practice. We also include model assessment and discussion about its possible practical applications and some ideas for further research.
The main contributions of this paper can be summarized as follows:

- Introducing a novel operations management planning model by integrating resource procurement, goods transformation, resources restrictions, market conditions, time restrictions and quality requirements into a multi-plant, multi-product and multi-period production network.
- Integrating linear programming measures into optimization of production process with the stress on cost minimization throughout the process.
- Application of proposed model providing real-life data and model verification.

1. Motivation for the study

Nowadays the global economic crisis deteriorates the business environment and makes it more difficult for enterprises to manage. Therefore enterprises must learn to adapt and make an effort to secure an effective and promising development in these constantly changing conditions. Therefore managers pay more attention to improving and optimizing possibilities, which would help their enterprise not only to survive but also to best fulfill their goals. Application of linear programming and process modelling are just two of these possibilities.

Linear programming was developed in early 50s simultaneously by Russian mathematicians Kantovich and Gavurin and American mathematician Dantzing. The most breakthrough discovery was Dantzing's creation of the Simplex method. Therefore he is referred to as "the father of linear programming". This method has been developed and evolved ever since mostly due to new technological inventions and wide-spread use of computers and other IT tools. Throughout almost 70 years of its usage as a method of optimization in enterprises linear programming has been applied in order to model many different types of enterprise's processes. This operations research method enables the transformation of real economic operations and processes into mathematical models.

Sarker, Newton (2008), Buresh-Openheim, Davis, Impagliazzo (2011) and Baker (2011) evaluated the advantages and disadvantages of the linear programming utilization. They both consider the possibility of applying these methods for the long-term production planning to be the most significant advantage. Other advantages include the relative accuracy of these methods for the needs of certain enterprises. The use of linear programming assumes the creation of the linear objective function which describes the problem as closely as possible. The variables also enable their modeling as closely to the conditions in the enterprise as possible. One of the main disadvantages of linear programming utilization is the fact that sometimes he linear function may not be the best option to model the processes and the situation may arise when enterprise would have to resort to other non-linear methods of the operations research. Despite of this fact, the advantages of linear programming utilization in enterprises are far greater and more significant. The application of these methods can help enterprises solve many different problems.

Linear programming is a powerful mathematical technique that can be used to solve various problems in managerial practice. Operations research is a management discipline which deals with optimizing problems in enterprises (Rader, 2010). Therefore the purpose of linear programming utilization in managerial practice is to solve linear optimizing problems. Many authors describe various possible utilizations of these methods in management. Some of the most common uses of linear programming are:

- max-cut tasks (Avis, Umemoto, 2003; Lodi, Monaci, 2003);
- allocation problems such as life cycle assessment (Azapagic, Clift, 1998); adaptability in agroforestry (Bertomeu, Bertomeu, Gimenez, 2006); staff training (Fagoyingo, Ajibode, 2010);
- mixing problems (Banks, 1979; Eiselt, Sandblom, 2010);
- routing optimization and transportation problems (Gass, 2010; Bley, 2011);
- scheduling: employees' work shifts (Baker, 2011); time minimizing problems (Floudas, Lin, 2005); draw control (Guest et al., 2012);
- financial management (Zemánková, Komorníková, 2008; Weber, 2009).

Moreover we cannot limit the possibilities of linear programming utilization only to the options described above. The limitations of its utilization are just in terms of needs of individual enterprises. Therefore we can state that linear programming can be used to provide solutions to a variety of different problems which enterprises encounter. Vanichchinchai, Igel (2011) provide evidence to this theory. They describe several other possibilities of how linear programming can improve enterprises' processes, specifically the decision-making process.

2. Linear programming model of production process optimization

Following the theoretical background and summarized knowledge we can create the linear programming integrated model of production process. This proposed model is based on linear programming and therefore it consists of objective function and related restrictions. The objective of the model is to minimize costs throughout production process. Restrictions focus on resource procurement

costs, fixed and variable production costs, inventory costs, transportation costs, market conditions, availability of resources and enterprise's production capacities.
Linear programming model is formulated as follows:

A_{nt}	availability of plant n at time t, its ability to produce, <0;1>
AR_{ri}	amount of resource r available
BR_{irt}	purchase quantity of resource r at time t
CQ_{ir}	accepted level of quality parameter
CR_{irt}	purchase price of resources at time t
C_R	purchase price of resource r
$D_{i,t-1}$	unsatisfied demand from previous time period
D_{it}	demand of product i in time period t
E_{rt}	amount of resource available on the market
FPC_{nt}	fixed costs necessary for plant n to run in time t
G_i	goal / planned amount of product i produced
h	other resources
i	product
k_{mt}	number of production runs on machine m at time t
K_r	total amount of resource available to enterprise
L	set of teams of employees available to the enterprise for production of product i
M	set of machines available to the enterprise for production of product i
MKT_{nt}	max. time of resource available
MOT_{nt}	overtime capacity of resource available if necessary, reserve
MZK_{it}	maximal inventory capacity for product i in time period t
MZK_{rt}	maximal inventory capacity for resource r in time period t
n	plant
Q_{ir}	quality parameter
r_R	amount of resource r needed to produce certain amount of product i
S_{it}	supply of product i already on the market
ST_m	machine set up time - time to start it, program it…
t	time
t_{il}	time team l needs to finish one unit of product i
t_{im}	time quota of product i processed on machine m
T_L	total labor working hours of team of employees
T_m	total number of machine working hours
u	transport method
UC_{inut}	transport costs
UK_{it}	total transport capacity in time t
UX_{it}	set of produced products assigned to transportation
VK_{nt}	production capacity of plant n at time t for product i
v_{nt}	1-0 function, 1 if plant n is opened in time period t, 0 otherwise
VPC_{inrt}	variable production costs
VT_{int}	average production time of product i
X_i	quantity of product i produced
X_{il}	amount of product i produced by team l
X_{im}	number of products i produced on machine m
z	inventory
ZC_{it}	inventory costs of holding one unit of product i during time t
ZC_{rt}	inventory costs of holding one unit of resource r during time period t

ZX_{it} amount of inventory of product i in time period t
Z_{rt} amount of inventory of resource r in time period t
δ_m probability that machine m runs normally

Minimize $z = \sum_i^I \sum_n^N \sum_r^R \sum_t^T (VPC_{inrt} \times X_{inrt} + CR_{nrt} \times BR_{nrt}) + \sum_i^I \sum_n^N \sum_t^T (ZC_{int} \times Z_{int}) +$
$+ \sum_r^R \sum_n^N \sum_t^T (ZC_{rnt} \times Z_{rnt}) + \sum_i^I \sum_n^N \sum_u^U \sum_t^T (UC_{inut} \times UX_{inut}) +$
$+ \sum_n^N \sum_t^T (FPC_{nt} \times v_{nt}) \; \forall \; i, n, r, u, t$ (1)

Subject to

$\sum_i^I (X_i \times AR_{ri}) \leq K_r \forall r$ (2)
$R \in \{L; M; H\}; R \in Z; R \geq 0$ (3)
$K_r = Z_r + E_r \forall r$ (4)
$\sum_i^I (r_R \times x_i) \times c_R = CR_{rt} \forall r$ (5)
$\sum_n^N (v_{nt} \times A_{nt} \times VK_{nt}) - \sum_n^N X_{int} \geq 0 \; \forall t$ (6)
$\sum_i^I (X_{il} \times t_{il}) \leq T_L \;\; \forall l$ (7)
$X_{it} = G_{it} \; \forall \; i, t$ (8)
$\sum_i^I (X_{im} \times t_{im} \div \delta_m + ST_m) \leq T_m \;\; \forall m$ (9)
$\sum_i^I \left(Z_{it} + ZX_{i,t-1}\right) \leq MZK_{it} \;\; \forall t$ (10)
$\sum_r^R \left(Z_{rt} + Z_{r,t-1}\right) \leq MZK_{rt} \forall t$ (11)
$ZX_{int} = ZX_{in,t-1} + X_{int} + UX_{int} \;\; \forall i, n, t$ (12)
$\sum_n^N X_{int} \geq \left(D_{it} + D_{i,t-1} - S_{it}\right) \forall i, t$ (13)
$\sum_n^N (X_{inr} \times VT_{inr}) \leq MKT_{nr} + MOT_{nr} \;\; \forall i, r$ (14)
$T_m \geq (k_m \times t_{im}) \; \forall m, i$ (15)
$Q_{ir} \times R_i \geq CQ_{ir} \times R_i \,, \forall r$ (16)
$\sum_i^I \sum_n^N UX_{it} \leq UK_{it} \;\; \forall u, t$ (17)
$X_i \geq 0; \; X_i \in Z$ (18)

Equation (1) represents the objective function of the linear programming model. Its main goal is to minimize most the costs any production enterprise has to deal with. The first part considers the variable production costs for current level of produced product including the resource procurement costs. The second and the third parts consider the inventory costs for both resources and final products. Transportation costs are also included. The final part of function represents the fixed production costs for each of the operable plants.

The first condition in the model (2) considers the availability of resources. The left part of inequality represents the amount of resources necessary to produce certain amount of product. This amount cannot be greater than total amount of resource available to enterprise. In this model the term resource does not only include raw material, but it also includes machinery, labor factor and other (3). For the needs of practical applications of model resource can only be nonnegative integer numbers. The third condition (4) specifies total amount of resource available to enterprise. It consists of inventory of resource and of the amount of resource available on the market. These conditions have to be met for all resources used by production enterprise.

Equation (5) represents the purchase costs of particular resources which are needed in order to produce all types of products. This condition also considers the amount of resource necessary for successful completion of production process.

Inequality (6) controls the production capacity of enterprise's plants. It ensures maintaining sustainable levels of production which are in accordance with the limits of capacity restrictions of plant. This condition needs to be met for every production facility. It considers also the fact whether the plant is opened and capable to operate normally in a given time period.

Next condition (7) considers the time restrictions for labor force. In this model we work with an assumption that each product is not produced by a single employee, but by a team of them. Therefore the equations describes the labor time of work teams. The amount of product produced by a single work team is multiplied by time it takes this team to produce one product. This total labor time of one team cannot be

higher than the total labor working hours of particular team of employees. This condition has to be adhered for every labor team of employees and during the production of all types of products.
Equation (8) establishes the fulfillment of production plan. In an ideal case total amount of product produced as an outcome of a certain production process during the considered time period should be equal to the amount of produced product described in plan. This condition should be considered in terms of production of all types of products and for all time periods set in the plan.

The next condition (9) represents the time restrictions of machines involved in the production process. This inequality considers the time quota of product processed on a machine and the amount of product produced on this machine. It also includes the possibility that the machine does not operate correctly or without failure. Time restrictions also involve the set up and start up time of each machine. All these time requirements during the production process have to be lower than total number of machine working hours during a certain time period. This condition has to be met for every machine involved in the production process regardless of its location.

Equations (10) and (11) describe the capacity restrictions on inventory of resources and final products. The amount of inventory of a certain product cannot be higher than the maximal capacity of storage. This involves all products which are produced and stored by the enterprise. Similarly this condition applies to the inventory of resources, especially raw material enterprise needs as an input of its production process. Both of these restrictions take into account also the inventory from previous time period which has not yet been consumed or shipped out. In a case enterprise uses one the production systems with no inventory policy (such as JIT for example), condition (11) is irrelevant. Equation (12) focuses on specifying the inventory of certain product in enterprise's storage places.

Equation (13) considers the limitations set on the enterprise by conditions on the market, in particular the demand of each product. The amount of produced product should be larger than the level of current demand and level of the unsatisfied demand from previous time period, but minus the supply of given product already on the market.

The next condition (14) establishes limitations for time capacity of resources. The average production time of one product multiplied by the amount of product produces has to be lower than maximal time of resource available including overtime capacity of resource available to the enterprise if necessary. This limitation has to be met for all resources and during the production of all products.

Inequality (15) controls the operating time of machines. Number of production runs on machine during the time period multiplied by the time quota of one product processed on machine has to be lower than total number of machine working hours during the same time period. This condition has to be met for all machines involved in production process in all production plants. This condition also considers the fact that each product could have different production time.
Equation (16) represents the general quality restrictions for the structure of produced product. A certain percentage of quality parameter of resource has to obtained in the final product. Its level is set In the Bill of Material and also in production plan of enterprise. This condition also works as foundation for determining the correct amount of resource needed to produce each product.

Transportation capacity is controlled by Equation (17). The amount of products designated for transport at the current time cannot be greater than the capacity of all transportation vehicles. This condition has to be met for all produced products at all facilities.

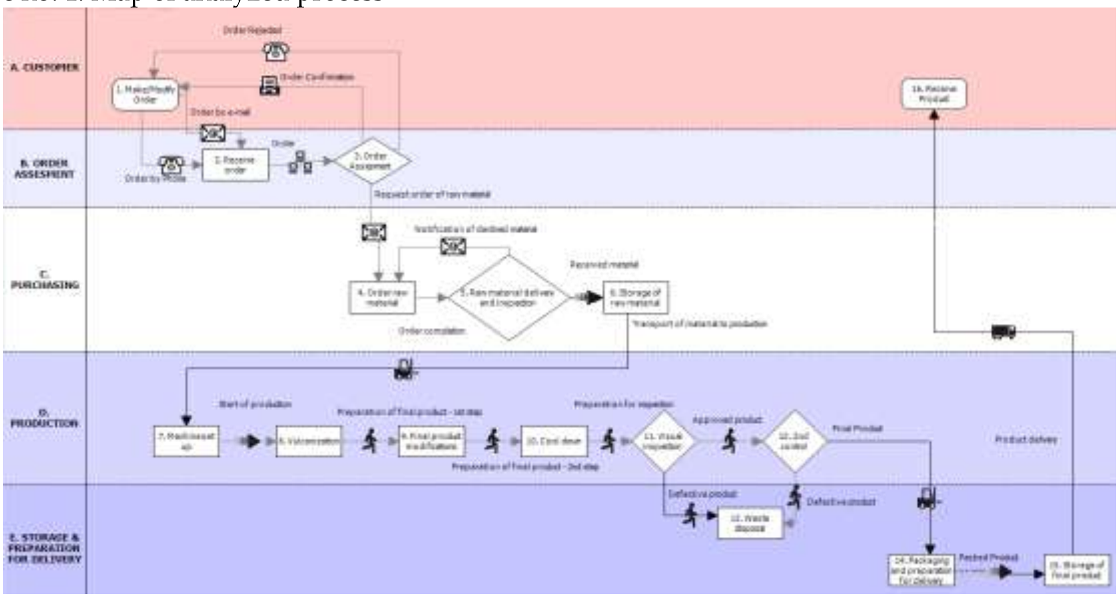
Lastly for the needs of practical applications of described model the amount of product produced by enterprise as an output of its production process has to be expressed as a nonnegative integer number (18).

3. Practical application of model on a real-life production process

We can now apply the proposed model on real-life production process. The goal is to both verify created model and to discover its limitations. We selected a medium-sized enterprise in order to analyze possible applications of created model. This enterprise is located in northern part of Slovak republic. It has approximately 110 employees and produces three different product types – rubber components, plastic hoses and concrete products. Therefore there are three different production processes in this enterprise which share space and resources. For our needs we chose the process of rubber components production for analysis and consequently for its optimization.

The first stage of any process optimization is to analyze selected process. We created a map of analyzed process in order to better understand it using the QPR Process Guide software (Picture no. 1).

Picture no. 1: Map of analyzed process



Source: Own elaboration.

Analyzed process can be divided into five main groups of activities – organizational units. This process starts when the customer makes a new order or notifies the enterprise about a specific modification of already submitted order. This new order or its modification is received by enterprise's production manager who is also in charge of assessing this order in terms of enterprise's production capacities. Customer is consequently notified whether the order is accepted or declined. Based on this decision the raw material necessary for new production is established. If it is necessary new order of raw material is made. This material is inspected upon arrival to the enterprise. If the inspection proves that the qualities of material are according to requirements, material is stored in enterprise's storage area under predetermined conditions. The production itself starts by setting up machine parameters. Then the vulcanization can be performed. Worker operating the machine eliminates any additional parts and he also needs to check the rubber hose according to predetermined parameters. If the product passes this visual inspection it is transported to the second machine which performs the second, more thorough inspection with stress on the structure of product and its chemical characteristics. Defective products and additional rubber parts are considered a waste and are eliminated. Products that pass both inspections can be transported to storage area and packed into badges of 1000 pieces. These are correctly labeled and ready for delivery to the customer. The delivery is performed on customer's expanses.

During the analysis of this production process we also collected specific data related to production indicators as well as the parameters set in our model. These parameters can have values as shows in Table no. 1.

Table no. 1 Values in parameters in linear programming model

A_{nt}	1	r	1
AR_{ri}	unlimited/undefined	S_{it}	undefined - enterprise is sole supplier on the relevant market
BR_{irt}	13370 kg/year	ST_m	11,67 minutes/day
CQ_{ir}; Q_{ir}	undefined	$t_{il}=t_{im}$	280 s/6 pieces
CR_{irt}	2,33 €/kg	T_L	1140 hours/month
$D_{i,t-1}$	10000 pieces/month	T_m	24 hours/ 24 hours
D_{it}	38000 pieces/month	u	6

E_{rt}	unlimited/undefined	UC_{inut}	1116,77 €/month/1 vehicle
FPC_{nt}	16533,44 €/month	UK_{it}	1000 pieces/5 minutes
G_i	650 pieces/ 450 minutes	UX_{it}	38000 pieces/month
h1 fuel	14571,20 €/year	VK_{nt}	340000 pieces/month
i	1	v_{nt}	16 hours/day = 1; 8 hours/day=0
k_{mt}	1/220s	VPC_{inrt}	$=CR_{int}+C_L$
K_r	1625 kg/month	VT_{int}	1000 pieces/hour
L	19 workers	X_i	33923 pieces/month
M_1	2 available	X_{il}	2000 pieces/2 workers/1 hour
M_2	1 available	X_{im}	6 pieces/280 s
MKT_{nt}	80 hours/month	Zcit=ZCrt	254,16 €/month
MOT_{nt}	12 hours/month/worker	ZX_{it}	200000 pieces/month
MZK_{it}	200000 pieces/month	Z_{rt}	15000 kg/month
MZK_{rt}	14000 kg	δ_{m1}	8 hours/month
n	1	δ_{m2}	30 minutes/month

Source: Own elaboration.

Implementing these values we modified proposed model and assessed it according to the real-life data in enterprise's practice. This way we not only managed to discover the critical points of the production process, but moreover we also managed to assess the proposed model based on its real-life application.

Discussions and conclusions

This paper presents an integrated linear programming model of production process with cost minimization as its main objective. We chose linear programming as a modeling tool because of its generality. It is a well-known fact that the majority of problems in practice of production enterprises can be expressed as linear programming tasks. Our goal was to develop a model of production process which would be flexible in its nature and would also be able to ensure achievement of its main objective. Integrating measures to ensure cost minimization into operations management is not a new trend, but it is especially important nowadays since the market conditions are changing rapidly and the whole economy is becoming very uncertain. Cost minimization is becoming one of the key survival tools of the majority of enterprises. It can also be an effective measure developed in order to deal with both inner and outer uncertainties.

The proposed linear programming model of production process is quite general in its nature which makes it possible to apply it on any production process. It can be modified to make it more suitable. An example of possible modifications was also described in this paper in a form of a case study of real-life production process optimization.

Proposed model described above considers following facts:

- Enterprise produces multiple types of products, which are not homogeneous in their nature and structure;
- Production process takes place during multiple time periods;
- Transportation of resources and products;
- Enterprise uses multiple types of inputs – raw material, labor, machinery etc. which enter the production process;
- Machinery – its production capacity, the duration of production runs and production process as whole including the set up times of machines;

- Enterprise has multiple production plants; each with its own fixed production costs, possibilities to operate properly and different production capacity;
- Market – the supply and demand of each product and the unsatisfied demand from previous time period;
- Market of production factors – different prices and availabilities of each factor;
- Enterprise's production plan, which in the ideal case should be adhered;
- Inventory – storage capacity for both final products and raw materials, costs of holding a certain amount of inventory of all products and resources.

Research limitations and direction for further research

Proposed model is general in its nature, which makes it possible to apply on any type of production process. Its main objective is to minimize production enterprise's costs and to make production process more flexible. This model is based on linear programming. The generality of model also brings many possibilities how to modify it and make it more suitable for a particular enterprise and its production process. These modifications also provide ideas for future research of the subject. The few of the most significant possibilities of modification are as follows:

- Time approach – a more dynamic model can also consider the variations from average production time and enable to divide the production process into various stages;
- Some parameters cannot be described by a certain number. Therefore a fuzzy approach should be considered, since it enables certain parameters to be expressed as interval. This approach can be more accurate especially in terms of variables which are difficult to express or their values can only be determinate from data originated in previous time periods;
- Model does con consider the distribution plan in detail. Therefore one of the possible modifications can involve optimization of enterprise's distribution plans and/or route optimization. Linear programming would be an effective tool in this case;
- Closer look into fixed production costs and their optimization. Some of the factors which influence their level can be evaluated;
- Resource allocation and optimization;
- Model does not consider the unfinished products in enterprise, their storage, transportation and so on;
- One of the other unexplored areas are employees. Model does not consider specialization of employees. All teams of them are universal and there are no restrictions for substitutions. This fact is considered in the objective function, where the wage differentiation is not considered. Labor costs are expressed for whole work team together.

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